Methods in Analysis: The Riesz-Thorin interpolation theorem

In this notes we give a proof of the Riesz-Thorin interpolation theorem. This theorem can be used, for example, to prove the Hausdorff-Young inequality, which establishes that the Fourier transform can be extended in a unique way as a continuous linear map $\hat{\cdot} : L^p \to L^{p'}$ for all $1 \leq p \leq 2$. Here and throughout these notes, we use the notation $L^p$ to indicate $L^p(\mathbb{R}^n)$ and prime indices like $p'$ will always indicate the dual of the index $p$, defined by the condition $1/p + 1/p' = 1$.

**Theorem 1.** Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $\theta \in (0, 1)$. Define $1 \leq p, q \leq \infty$ by

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.
\]

If $T$ is a linear map with

\[
T : L^{p_0} \to L^{q_0}, \quad ||T||_{L^{p_0} \to L^{q_0}} = N_0
\]

\[
T : L^{p_1} \to L^{q_1}, \quad ||T||_{L^{p_1} \to L^{q_1}} = N_1
\]

then we have

\[
||Tf||_q \leq N_0^{1-\theta} N_1^\theta ||f||_p
\]

for all $f \in L^{p_0} \cap L^{p_1}$. Hence $T$ extends uniquely as a continuous map from $L^p$ into $L^q$, with $||T||_{L^p \to L^q} \leq N_0^{1-\theta} N_1^\theta$.

**Remark 2.** The two maps $T : L^{p_0} \to L^{q_0}$ with $||T||_{L^{p_0} \to L^{q_0}} = N_0$ and $T : L^{p_1} \to L^{q_1}$ with $||T||_{L^{p_1} \to L^{q_1}} = N_1$ are, strictly speaking, two different maps that agree on $L^{p_0} \cap L^{p_1}$. For $1 \leq p_0, p_1 < \infty$, this is equivalent to saying that there exists a map $T : L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ with

\[
\sup_{f \in L^{p_0} \cap L^{p_1}, ||f||_{p_0} \leq 1} ||Tf||_{q_0} = N_0, \quad \text{and} \quad \sup_{f \in L^{p_0} \cap L^{p_1}, ||f||_{p_1} \leq 1} ||Tf||_{q_1} = N_1.
\]

In fact, if this is true (and if $1 \leq p_0, p_1 < \infty$), $T$ can be uniquely extended to continuous linear maps $T_0 : L^{p_0} \to L^{q_0}, T_1 : L^{p_1} \to L^{q_1}$ such that $||T_0|| = N_0$, $||T_1|| = N_1$, and $T_0 f = T_1 f = T f$ for all $f \in L^{p_0} \cap L^{p_1}$.

**Remark 3.** If $f \in L^{p_0} \cap L^{p_1}$ and $p$ is defined by $1/p = (1-\theta)/p_0 + \theta/p_1$, then $f \in L^p$ and $||f||_p \leq ||f||_{p_0}^{1-\theta} ||f||_{p_1}^\theta$. This fact can be proven by Hölder’s inequality.

**Remark 4.** If we associate a point $(1/p, 1/q)$ to every pair of indices $1 \leq p, q \leq \infty$, the theorem states that, if $T$ defines a continuous map from $L^{p_0}$ into $L^{q_0}$ and from $L^{p_1}$ into $L^{q_1}$, then it also maps continuously $L^p$ into $L^q$ for all $p, q$ such that the point $(1/p, 1/q)$ lies in the segment between $(1/p_0, 1/q_0)$ (the convex hull of the two points $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1)$).

The proof of the Riesz-Thorin interpolation theorem is based on the following simple Lemma.

**Lemma 5** (Hadamard three line lemma). Let $S = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \}$ and $F : S \to \mathbb{C}$ be bounded and continuous on $S$ and analytic on the interior $S_0$ of $S$. Let $M_\theta = \sup_{\theta \in \mathbb{R}} |F(\theta + iy)|$. Then we have $M_\theta \leq M_0^{1-\theta} M_1^\theta$ for all $\theta \in [0, 1]$. 


Proof. Without loss of generality, we can assume that $M_0 = M_1 = 1$. Otherwise we replace $F$ by the function $\phi : S \to \mathbb{C}$ defined by $\phi(z) = F(z)/(M_0^{-1}M_1^*)$. By the assumptions on $F$, it follows that $\phi$ is continuous and bounded on $S$ (because $|\phi(z)| \leq |F(z)|/(M_0^{-1}\Re z M_1^* \Re z) \leq |F(z)|/(\min(1, M_0) \min(1, M_1))$ and $M_0, M_1$ cannot be zero) and analytic on $S_0$, with $\sup_{\Re z = 0} |\phi(z)| = \sup_{\Re z = 1} |\phi(z)| = 1$. Hence, we can assume that

$$\sup_{\Re z = 0, 1} |F(z)| = 1$$

and, under this assumption, we need to show that

$$\sup_{z \in S} |F(z)| \leq 1.$$

To this end, we define the sequence $F_n(z) = F(z)e^{z^2/n}e^{-1/n}$ and we observe that $|F_n(z)| \leq |F(z)|$ for all $z \in S$; in particular, $\sup_{z = 0, 1} |F_n(z)| \leq 1$. Moreover, $F_n(z)$ is analytic in $S_0$ for all $n \geq 1$, and $|F_n(x + iy)| \to 0$, as $|y| \to \infty$, for every fixed $n$, uniformly in $x$. Hence, we obtain that

$$\sup_{z \in S} |F_n(z)| \leq 1$$

for every $n \geq 1$, because analytic functions attain their maximum and minimum on the boundary of any compact set (consider the compact domain $K = \{z : \Im z \leq \kappa, 0 \leq \Re z \leq 1\}$, where $\kappa$ is so large that $|F_n(x + iy)| \leq 1$ for all $|y| \geq \kappa$, and $x \in [0, 1]$). Since $|F_n(z)| \to |F(z)|$ as $n \to \infty$, it follows that $|F(z)| \leq 1$ for all $z \in S$.

Proof of the Riesz-Thorin interpolation theorem. We consider the case $p < \infty$ and $q > 1$ only. For $1 \leq p_0, p_1 < \infty$, we know that continuous compactly supported functions are dense in $L^{p_0} \cap L^{p_1}$ (with respect to the norm $\|\cdot\|_{L^{p_0} \cap L^{p_1}} = \|\cdot\|_{p_0} + \|\cdot\|_{p_1}$). Since compactly supported continuous functions are uniformly continuous, it follows that they can be approximated by compactly supported step-functions taking only finitely many values (divide the support into sufficiently small cubes, and replace, within each cube, the function by its average). Hence, it is enough to show that

$$\|Tf\|_q \leq N_0^{1-\theta}N_1^\theta\|f\|_p$$

(1)

for every compactly supported step function $f$. In fact, if we assume (1) then, for a general $f \in L^{p_0} \cap L^{p_1}$ (1 \leq p_0, p_1 < \infty), we can find a sequence $f_n$ of compactly supported step functions such that $f_n \to f$ in $L^{p_0} \cap L^{p_1}$, which implies that $f_n \to f$ in $L^{p_0}$ and $f_n \to f$ in $L^{p_1}$. Thus, using Remark 3, we find

$$\|Tf\|_q \leq \|T(f - f_n)\|_q + \|Tf_n\|_q \leq \|T(f - f_n)\|_{q_0}^{1-\theta}\|T(f - f_n)\|_{q_1}^\theta + N_0^{1-\theta}N_1^\theta\|f_n\|_p \leq \|T(f - f_n)\|_{q_0}^{1-\theta}\|T(f - f_n)\|_{q_1}^\theta + N_0^{1-\theta}N_1^\theta\|f_n\|_p + N_0^{1-\theta}N_1^\theta\|f\|_p$$

(2)

and since the second term in the parenthesis tends to zero, as $n \to \infty$, we conclude that $\|Tf\|_q \leq N_0^{1-\theta}N_1^\theta\|f\|_p$ for all $f \in L^{p_0} \cap L^{p_1}$.

Note that also in the case $1 \leq p_0 < p_1 = \infty$ (and in the symmetric case $1 \leq p_1 < p_0 = \infty$) it is enough to show (1) for compactly supported step functions; in fact, in this case, it is still possible to find a sequence $f_n$ of compactly supported step functions so that $\|f_n - f\|_{p_0} \to 0$ as $n \to \infty$ and
\[ \|f_n - f\|_\infty \leq 2\|f\|_\infty. \] Since \( \theta < 1 \) (by the assumption \( p_0 < p_1 \)), this is still enough to show that the second term in the parenthesis on the r.h.s. of (2) vanishes, as \( n \to \infty \) (observe that the case \( p_0 = p_1 = \infty \) is excluded by the condition \( p < \infty \)).

In order to show (1), we use that
\[ \|v\|_q = \sup_{\|g\|_{q'} \leq 1} \left| \int g v \right| \]
where \( q' \) is the dual index to \( q \). Using the density of the compactly supported step functions in \( L^{q_0} \cap L^{q_1} \), we see that it is enough to show that
\[ \left| \int g Tf \right| \leq N_0^{1-\theta} N_1^\theta \] for all compactly supported step functions \( f, g \) with \( \|f\|_p = \|g\|_{q'} = 1 \).

To prove (3), we write \( f = \sum_{j=1}^n a_j \chi_{A_j} \), where \( a_1, \ldots, a_n \in \mathbb{C}, A_1, \ldots, A_n \) are measurable, pairwise disjoint subsets of \( \mathbb{R}^n \), and \( \chi_A \) denotes the characteristic function of the set \( A \). Analogously, we write \( g = \sum_{\ell=1}^m b_\ell \chi_{B_\ell} \), with \( b_1, \ldots, b_m \in \mathbb{C} \) and \( B_1, \ldots, B_m \) measurable and pairwise disjoint.

Note that
\[ \|f\|_p = \sum_{j=1}^n |a_j|^p \lambda(A_j) = 1, \quad \|g\|_{q'} = \sum_{\ell=1}^m |b_\ell|^{q'} \lambda(B_\ell) \] (4)
where \( \lambda(.) \) denotes Lebesgue measure.

For \( z \in \mathbb{C} \) we define the functions \( p(z), q'(z) \) by
\[ \frac{1}{p(z)} = \frac{1}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1}{q'_0} + \frac{z}{q'_1} \]
Observe that \( p(0) = p_0, p(1) = p_1, p(\theta) = p \), and \( q'(0) = q'_0, q'(1) = q'_1, q'(\theta) = q' \). Using \( p(z) \) and \( q'(z) \) we define the functions
\[ f_z(x) = |f(x)|^{\frac{p(z)}{p}} \frac{f(x)}{|f(x)|}, \quad g_z(x) = |g(x)|^{\frac{q'(z)}{\theta}} \frac{g(x)}{|g(x)|} \]
with the convention that \( f(x)/|f(x)| = 0 \) if \( f(x) = 0 \). For every \( z \in \mathbb{C} \), \( f_z \) and \( g_z \) are compactly supported step functions (of \( x \)) given by
\[ f_z(x) = \sum_{j=1}^n |a_j|^{\frac{p(z)}{p}} \frac{a_j}{|a_j|} \chi_{A_j}, \quad g_z(x) = \sum_{\ell=1}^m |b_\ell|^{\frac{q'(z)}{q'}} \frac{b_\ell}{|b_\ell|} \chi_{B_\ell} \] (5)
In particular, \( f_z \in L^{p_0} \cap L^{p_1} \) for every \( z \in \mathbb{C} \) and therefore \( T f_z \in L^{q_0} \cap L^{q_1} \) is well-defined and can be integrated against the function \( g_z \in L^{q'_0} \cap L^{q'_1} \). We set
\[ F(z) = \int g_z T f_z. \]
Using (5), we obtain the representation
\[ F(z) = \sum_{j=1}^n \sum_{\ell=1}^m |a_j|^{\frac{p(z)}{p}} \frac{a_j}{|a_j|} |b_\ell|^{\frac{q'(z)}{q'}} \frac{b_\ell}{|b_\ell|} \int_{B_\ell} T \chi_{A_j}. \]
We see that $F$, as a function of $z$, is a linear combination of terms of the form $\gamma z$, for appropriate positive $\gamma$’s. Hence $F$ is bounded and continuous on the strip $S = \{ z \in \mathbb{C} : 0 \leq \text{Re } z \leq 1 \}$ (because $|\gamma z| = \gamma \text{Re } (z) \leq \max(1, \gamma)$ for every $z \in S$) and it is analytic on the interior $S_0$ of $S$. Therefore we can apply Lemma 5 to $F(z)$. It implies that

$$M_\theta = \sup_{y \in \mathbb{R}} |F(\theta + iy)| \leq M_1^{1-\theta} M_1^\theta$$

and therefore, since for $z = \theta$, we have $f_{z=\theta} = f$ and $g_{z=\theta} = g$, we find

$$\left| \int g T f \right| = |F(\theta)| \leq M_\theta \leq M_1^{1-\theta} M_1^\theta.$$  \hspace{1cm} (6)

We still have to compute $M_0$ and $M_1$. To compute $M_0$ we estimate

$$|F(iy)| = \left| \int g_{iy} T f_{iy} \right| \leq \| T f_{iy} \|_q \| g_{iy} \|_{q_0'} \leq N_0 \| f_{iy} \|_p \| g_{iy} \|_{q_0'}.$$

By definition, we have

$$\| f_{iy} \|_p^{p_0} = \sum_{j=1}^n |a_j|^{p_0} \lambda(A_j) = \sum_{j=1}^n |a_j|^p \lambda(A_j) = \| f \|_p^p = 1$$

and

$$\| g_{iy} \|_{q_0'}^{q_0'} = \sum_{\ell=1}^m |b_{\ell'}|^{q_0'} \lambda(B_{\ell'}) = \sum_{j=1}^m |b_{\ell'}|^q \lambda(B_{\ell'}) = \| g \|_{q'}^q = 1$$

and therefore $|F(iy)| \leq N_0$ for all $y \in \mathbb{R}$, which implies that $M_0 \leq N_0$. Similarly, to bound $M_1$, we compute

$$|F(1 + iy)| = \left| \int g_{1+iy} T f_{1+iy} \right| \leq \| T f_{1+iy} \|_{q_1} \| g_{1+iy} \|_{q_1'} \leq N_1 \| f_{1+iy} \|_{p_1} \| g_{1+iy} \|_{q_1'}.$$

where

$$\| f_{1+iy} \|_{p_1}^{p_1} = \sum_{j=1}^n |a_j|^{p_1} \lambda(A_j) = \sum_{j=1}^n |a_j|^p \lambda(A_j) = \| f \|_p^p = 1$$

and

$$\| g_{1+iy} \|_{q_1'}^{q_1'} = \sum_{\ell=1}^m |b_{\ell'}|^{q_1'} \lambda(B_{\ell'}) = \sum_{j=1}^m |b_{\ell'}|^q \lambda(B_{\ell'}) = \| g \|_{q'}^q = 1$$

which implies that $M_1 \leq N_1$ and, by (6), completes the proof of (3). \hfill \Box